4. $\hat{L}^2 = -\hbar^2 \Lambda^2$

$$= -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right]$$

(a) $\hat{L}^2 \left[\frac{1}{(4\pi)^{1/2}} \right] = 0$ because the fn. is independent of θ and ϕ

(b) $$\hat{L}^2 \left(\sqrt{\frac{3}{4\pi}} \cos\theta \right) = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \left\{ \frac{\partial}{\partial\theta} \cos\theta \right\} + \overset{\text{no } \phi \text{ depende}}{0} \right] \sqrt{\frac{3}{4\pi}}$$

$$= -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} (-\sin^2\theta) \right] \cdot \sqrt{\frac{3}{4\pi}}$$

$$= \hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin^2\theta \right] \cdot \sqrt{\frac{3}{4\pi}}$$

$$= \hbar^2 \cdot \frac{2\sin\theta\cos\theta}{\sin\theta} \sqrt{\frac{3}{4\pi}}$$

$$= 2\hbar^2 \sqrt{\frac{3}{4\pi}} \cos\theta$$

(c) $$\hat{L}^2 \left(\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi} \right)$$

$$= -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \left\{ \frac{\partial}{\partial\theta} \sin\theta e^{i\phi} \right\} + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \sin\theta e^{i\phi} \right] \sqrt{\frac{3}{8\pi}}$$

$$= -\hbar^2 \left[e^{i\phi} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left\{ \sin\theta\cos\theta \right\} + \frac{1}{\sin^2\theta} \sin\theta \, (i)^2 e^{i\phi} \right] \sqrt{\frac{3}{8\pi}}$$

$$= -\hbar^2 \sqrt{\frac{3}{8\pi}}\, e^{i\phi}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta) - \frac{1}{\sin\theta} \right]$$

$$= -\hbar^2 \sqrt{\frac{3}{8\pi}}\, e^{i\phi}\left(\frac{\cos^2\theta - \sin^2\theta}{\sin\theta} - \frac{1}{\sin\theta} \right)$$

$$= -\hbar^2 \sqrt{\frac{3}{8\pi}}\, e^{i\phi}\left[\frac{\cos^2\theta - \sin^2\theta - 1}{\sin\theta} \right]$$

$$= -\hbar^2 \sqrt{\frac{3}{8\pi}}\, e^{i\phi}\left[-\frac{2\sin^2\theta}{\sin\theta} \right]$$

$$= 2\hbar^2 \sqrt{\frac{3}{8\pi}}\, \sin\theta\, e^{i\phi}$$

Note: All these spherical harmonics are eigenfunctions of $\hat{L}^2$ and the eigenvalues are multiples of $\hbar^2$.

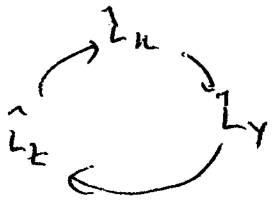
8. $\hat{L}_+ = \hat{L}_x + i\hat{L}_y \qquad \hat{L}_- = \hat{L}_x - i\hat{L}_y$

(a) $\hat{L}_+\hat{L}_- = [\hat{L}_x + i\hat{L}_y][\hat{L}_x - i\hat{L}_y]$

$= \hat{L}_x^2 - i\hat{L}_x\hat{L}_y + i\hat{L}_y\hat{L}_x - i^2\hat{L}_y^2$

$= \hat{L}_x^2 + \hat{L}_y^2 + i[\hat{L}_y\hat{L}_x - \hat{L}_x\hat{L}_y]$

$= \hat{L}_x^2 + \hat{L}_y^2 + i[\hat{L}_y, \hat{L}_x]$



$= \hat{L}^2 - \hat{L}_z^2 + i[-i\hbar\hat{L}_z]$

$= \hat{L}^2 - \hat{L}_z^2 + \hbar\hat{L}_z$

$\hat{L}_-\hat{L}_+ = [\hat{L}_x - i\hat{L}_y][\hat{L}_x + i\hat{L}_y]$

$= \hat{L}_x^2 + i\hat{L}_x\hat{L}_y - i\hat{L}_y\hat{L}_x - i^2\hat{L}_y^2$

$= \hat{L}_x^2 + \hat{L}_y^2 + i[\hat{L}_x, \hat{L}_y]$

$= \hat{L}^2 - \hat{L}_z^2 + i(i\hbar\hat{L}_z)$

$= \hat{L}^2 - \hat{L}_z^2 - \hbar\hat{L}_z$

$\underline{\hat{L}_+\hat{L}_- + \hat{L}_-\hat{L}_+ = 2(\hat{L}^2 - \hat{L}_z^2)}$

(b) Here $[\hat{L}_+, \hat{L}_-] = [\hat{L}_+\hat{L}_- - \hat{L}_-\hat{L}_+]$

From part (a) we have

$$\hat{L}_+\hat{L}_- - \hat{L}_-\hat{L}_+ = 2\hbar\hat{L}_z$$

$$\therefore \; 2\hbar\hat{L}_z Y_{lm_l}(\theta,\phi) = 2\hbar\, m_l\hbar\, Y_{lm_l}(\theta,\phi)$$

$$= \underline{2m_l\hbar^2 Y_{lm_l}(\theta,\phi)}$$

eigenvalue $= 2m_l\hbar^2$

m_l can take both +ve and -ve 'l' values

8 (c)

$$L_x = \frac{1}{2}(L_+ + L_-)$$

$$\therefore \langle L_x \rangle = \frac{1}{2}\left[\langle L_+\rangle + \langle L_-\rangle\right] \quad \cdots\cdots \quad (1)$$

$$\langle L_+\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; Y^*_{l,m}(\theta,\phi)\, L_+ \, Y_{l,m}(\theta,\phi)$$

$$= c_+ \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; Y^*_{l,m}(\theta,\phi)\, Y_{l,m+1}(\theta,\phi)$$

[using $L_+ Y_{l,m}(\theta,\phi) = c_+ Y_{l,}$

$$= c_+ \delta_{ll}\, \delta_{m,m+1} \qquad [\delta_{ll} = 1 \,;\; \text{but } \delta_{m,m+1} = 0]$$

$= 0$ (where we used the orthonormali property the spherical harmonics]

Similarly
$$\langle L_-\rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; Y^*_{l,m}(\theta,\phi)\, L_- \, Y_{l,m}(\theta,\phi)$$

$$= c_- \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \; \underbrace{Y^*_{l,m}(\theta,\phi)\, Y_{l,m-1}(\theta,\phi)}_{\text{as before these are orthogonal}}$$

$$= c_- \delta_{ll}\, \delta_{m,m-1} \qquad [\delta_{m,m-1} = 0 \text{ since } m \neq m-1]$$

$$= 0$$

$$\therefore \langle L_x \rangle = 0$$

9. $Y_1(\theta, \phi) = N_1 \sin^2\theta \cos\theta \, e^{\pm 2i\phi}$

Compare with the general equation

$$Y_{l m_l}(\theta, \phi) = N_{l m_l} \, \Theta(\theta) \, e^{i m_l \phi}$$

Hence $m_l = \pm 2$

To predict the value of 'l' $\rightarrow$ should always look at the maximum power of θ

Here $\Rightarrow$ $\sin^2\theta \cos\theta$ $\Rightarrow$ power of $\theta = 3$

Hence $l = 3$

$l = 3$; $m_l = \pm 2$

Similarly for $Y_2(\theta, \phi) = N_2 (5\cos^3\theta - 3\cos\theta)$

$m_l = 0$

$l = 3$ [maximum $\cos^3\theta$]

10. $l = 3$; there are '$2l+1$' m_l values

z projection $= 2\hbar$

Semi-angle of the cone is given by

$$\cos\theta = \frac{m_l \hbar}{\sqrt{l(l+1)}\,\hbar} = \frac{m_l}{\sqrt{l(l+1)}}$$

$$= \frac{2}{\sqrt{3.4}} = \frac{2}{2\sqrt{3}}.$$

$$= \frac{1}{\sqrt{3}}$$

11. $\mu^- = 206\, m_e$

$$\mu_\mu = \frac{m_e m_{\mu^-}}{m_e + m_{\mu^-}} = \frac{m_e \cdot 206 m_e}{m_e + 206 m_e} = \frac{206}{207} m_e$$

$$= \left(1 - \frac{1}{207}\right) m_e$$

$$a_\mu = \frac{4\pi\varepsilon_0 \hbar^2}{\mu e^2} = \frac{\hbar^2}{\mu e^2} \quad \text{(in cgs unit)}$$

$$a_\mu = \frac{4\pi\varepsilon_0 \hbar^2}{e^2} \cdot \frac{1}{\left(1 - \frac{1}{207}\right) m_e}$$

$$= \frac{4\pi\varepsilon_0 \hbar^2}{e^2} \cdot \frac{1}{\frac{206}{207} m_e} = \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2} \cdot \frac{207}{206}$$

$$= (a_0)_H \cdot \frac{207}{206}$$

$$E_n = -\left(\frac{\mu e^4}{32\pi^2 \varepsilon_0^2 \hbar^2}\right) \frac{1}{n^2}$$

also $$E = -\frac{Z^2 \mu e^4}{32\pi^2 \varepsilon_0^2 \hbar^2} \cdot \frac{1}{n^2}$$

$$= -\frac{\mu e^4}{8 h^2 \varepsilon_0^2} \cdot \frac{1}{n^2} \qquad Z = 1$$

$$E_{\text{H-like}} = -13.6 \frac{Z^2}{n^2}$$

For 1s, $n=1$

$$E = -\frac{\mu e^4}{8h^2\varepsilon_0^2}\cdot\frac{1}{1^2}$$

$$= -\frac{e^4 \cdot m_e}{8h^2\varepsilon_0^2}\cdot\frac{206}{207} = -(E_{1s})_H \frac{206}{207}$$

$$= -13.6 \cdot \frac{206}{207} \text{ eV}$$

using $m_e = 9.1 \times 10^{-31}$ kg

$e = 1.60 \times 10^{-19}$ C

$h = 6.626 \times 10^{-34}$ J s

$\varepsilon_0 = 8.85 \times 10^{-12}$ $C^2/N\text{-}m$

1eV = 2.18×10^{-18} J

For 2p, $$E = -\frac{13.6}{2^2}\cdot\frac{206}{207} \text{ eV}$$

For most probable radius, relate to problem 16